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Note on the Integration of a certain System of Differential Equations.

BY JOHN EIESLAND.

Professor Craig, in an article which appears in the present volume of this Journal and is entitled "Displacements depending on One, Two and Three Variables in a Space of Four Dimensions," deduces the following system of differential equations :

$$\left. \begin{aligned} \frac{d\alpha}{dt} &= p_{12}\beta - p_{13}\gamma + p_{14}\delta, \\ \frac{d\beta}{dt} &= -p_{12}\alpha + p_{23}\gamma + p_{24}\delta, \\ \frac{d\gamma}{dt} &= p_{13}\alpha - p_{23}\beta + p_{34}\delta, \\ \frac{d\delta}{dt} &= -p_{14}\alpha - p_{24}\beta - p_{34}\gamma, \end{aligned} \right\} \quad (1)$$

which are satisfied by the four groups of direction cosines a, b, c, d ; a', b', c', d' ; a'', b'', c'', d'' ; a''', b''', c''', d''' and admits of the integral

$$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = \text{const.}$$

We shall first consider the case where the constant differs from zero. By dividing the left side by a suitable constant, we may always suppose that

$$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 1. \quad (2)$$

We now employ the transformation

$$\alpha = \frac{\lambda}{\sqrt{x^2 + 1}}, \quad \beta = \frac{\mu}{\sqrt{x^2 + 1}}, \quad \gamma = \frac{\nu}{\sqrt{x^2 + 1}}, \quad \delta = \frac{1}{\sqrt{x^2 + 1}},$$

where $\kappa^2 = \lambda^2 + \mu^2 + \nu^2$. The system (1) now takes the form

$$\left. \begin{aligned} \frac{d\lambda}{dt} &= p_{14} + p_{12}\mu - p_{13}\nu + \lambda(p_{14}\lambda + p_{24}\mu + p_{34}\nu), \\ \frac{d\mu}{dt} &= p_{24} - p_{12}\lambda + p_{23}\nu + \mu(p_{14}\lambda + p_{24}\mu + p_{34}\nu), \\ \frac{d\nu}{dt} &= p_{34} + p_{13}\lambda - p_{23}\mu + \nu(p_{14}\lambda + p_{24}\mu + p_{34}\nu), \end{aligned} \right\} \quad (3)$$

which is a generalization of Ricatti's equation.

If now we consider λ, μ, ν as coordinates in ordinary space and t as time, this system defines an infinitesimal transformation :

$$\begin{aligned} Uf = & \{p_{14} + p_{12}\mu - p_{13}\nu + \lambda(p_{14}\lambda + p_{24}\mu + p_{34}\nu)\} \frac{\partial f}{\partial \lambda} \\ & + \{p_{24} - p_{12}\lambda + p_{23}\nu + \mu(p_{14}\lambda + p_{24}\mu + p_{34}\nu)\} \frac{\partial f}{\partial \mu} \\ & + \{p_{34} + p_{13}\lambda - p_{23}\mu + \nu(p_{14}\lambda + p_{24}\mu + p_{34}\nu)\} \frac{\partial f}{\partial \nu}, \end{aligned} \quad (4)$$

which is performed on the point λ, μ, ν in the element of time dt . A point λ, μ, ν is thus transformed from an initial position λ_0, μ_0, ν_0 into another general position λ, μ, ν by means of a projective transformation which changes with the time t . The general integral of (3) will therefore have the form of a finite projective transformation

$$\begin{aligned} \lambda &= \frac{a_1\lambda_0 + b_1\mu_0 + c_1\nu_0 + d_1}{a_4\lambda_0 + b_4\mu_0 + c_4\nu_0 + d_4}, & \mu &= \frac{a_2\lambda_0 + b_2\mu_0 + c_2\nu_0 + d_2}{a_4\lambda_0 + b_4\mu_0 + c_4\nu_0 + d_4}, \\ \nu &= \frac{a_3\lambda_0 + b_3\mu_0 + c_3\nu_0 + d_3}{a_4\lambda_0 + b_4\mu_0 + c_4\nu_0 + d_4}, \end{aligned} \quad (5)$$

where the coefficients a_i, b_i, c_i, d_i , ($i = 1, 2, 3, 4$) are functions of t , and λ_0, μ_0, ν_0 are the coordinates of the initial point and play the role of arbitrary constants of integration. It is easily seen that the coefficients a_i, b_i, c_i, d_i are particular solutions of the system (1). In fact, the general integrals of this are

$$\begin{aligned} \alpha &= k_1a_1 + k_2b_1 + k_3c_1 + k_4d_1, \\ \beta &= k_1a_2 + k_2b_2 + k_3c_2 + k_4d_2, \\ \gamma &= k_1a_3 + k_2b_3 + k_3c_3 + k_4d_3, \\ \delta &= k_1a_4 + k_2b_4 + k_3c_4 + k_4d_4, \end{aligned}$$

where $a_1, b_1, c_1, d_1; a_2, b_2, c_2, d_2; a_3, b_3, c_3, d_3; a_4, b_4, c_4, d_4$ form a set of four particular solutions of (1), and from this system of integrals we readily pass to (5) by taking into account the relations

$$\lambda = \frac{\alpha}{\delta}, \quad \mu = \frac{\beta}{\delta}, \quad \nu = \frac{\gamma}{\delta}.$$

Suppose now that we know a set of three particular solutions of the system (3). Let this be $\lambda_1, \lambda_2, \lambda_3; \mu_1, \mu_2, \mu_3; \nu_1, \nu_2, \nu_3$, the determinant of which we suppose different from zero. We are able to express the general system of integrals by means of these solutions.

An easy calculation will show that this system is

$$\left. \begin{aligned} \lambda &= \frac{\frac{\lambda_1}{\sqrt{\kappa_1^2+1}}\lambda_0 + \frac{\lambda_2}{\sqrt{\kappa_2^2+1}}\mu_0 + \frac{\lambda_3}{\sqrt{\kappa_3^2+1}}\nu_0 + \frac{\lambda_4}{\sqrt{\kappa_4^2+1}}}{\frac{1}{\sqrt{\kappa_1^2+1}}\lambda_0 + \frac{1}{\sqrt{\kappa_2^2+1}}\mu_0 + \frac{1}{\sqrt{\kappa_3^2+1}}\nu_0 + \frac{1}{\sqrt{\kappa_4^2+1}}}, \\ \mu &= \frac{\frac{\mu_1}{\sqrt{\kappa_1^2+1}}\lambda_0 + \frac{\mu_2}{\sqrt{\kappa_2^2+1}}\mu_0 + \frac{\mu_3}{\sqrt{\kappa_3^2+1}}\nu_0 + \frac{\mu_4}{\sqrt{\kappa_4^2+1}}}{\frac{1}{\sqrt{\kappa_1^2+1}}\lambda_0 + \frac{1}{\sqrt{\kappa_2^2+1}}\mu_0 + \frac{1}{\sqrt{\kappa_3^2+1}}\nu_0 + \frac{1}{\sqrt{\kappa_4^2+1}}}, \\ \nu &= \frac{\frac{\nu_1}{\sqrt{\kappa_1^2+1}}\lambda_0 + \frac{\nu_2}{\sqrt{\kappa_2^2+1}}\mu_0 + \frac{\nu_3}{\sqrt{\kappa_3^2+1}}\nu_0 + \frac{\nu_4}{\sqrt{\kappa_4^2+1}}}{\frac{1}{\sqrt{\kappa_1^2+1}}\lambda_0 + \frac{1}{\sqrt{\kappa_2^2+1}}\mu_0 + \frac{1}{\sqrt{\kappa_3^2+1}}\nu_0 + \frac{1}{\sqrt{\kappa_4^2+1}}}, \end{aligned} \right\} \quad (6)$$

where

$$\lambda_4 = \frac{\begin{vmatrix} \mu_1 & \nu_1 & 1 \\ \mu_2 & \nu_2 & 1 \\ \mu_3 & \nu_3 & 1 \end{vmatrix}}{\Delta}, \quad \mu_4 = \frac{\begin{vmatrix} \lambda_1 & \nu_1 & 1 \\ \lambda_2 & \nu_2 & 1 \\ \lambda_3 & \nu_3 & 1 \end{vmatrix}}{\Delta}, \quad \nu_4 = \frac{\begin{vmatrix} \lambda_1 & \mu_1 & 1 \\ \lambda_2 & \mu_2 & 1 \\ \lambda_3 & \mu_3 & 1 \end{vmatrix}}{\Delta}, \quad \Delta = \begin{vmatrix} \lambda_1 & \mu_1 & \nu_1 \\ \lambda_2 & \mu_2 & \nu_2 \\ \lambda_3 & \mu_3 & \nu_3 \end{vmatrix}$$

Returning now to the system (3), we will consider λ, μ, ν, t as coordinates in a four-dimensional space. An integral hypersurface may be represented by the equation

$$\nu = \rho\lambda + \sigma\mu + \tau, \quad (7)$$

ρ, σ , and τ being certain unknown functions of t . Differentiating and substituting the values of $\frac{d\lambda}{dt}, \frac{d\mu}{dt}, \frac{d\nu}{dt}$ given by (3), we get

$$\begin{aligned}
& p_{34} + p_{13}\lambda - p_{23}\mu + \nu(p_{14}\lambda + p_{24}\mu + p_{34}\nu) \\
& - \rho\{p_{14} + p_{12}\mu - p_{13}\mu + \lambda(p_{14}\lambda + p_{24}\mu + p_{34}\nu)\} \\
& - \sigma\{p_{24} - p_{12}\lambda + p_{23}\nu + \mu(p_{14}\lambda + p_{24}\mu + p_{34}\nu)\} \\
& - \lambda \frac{d\rho}{dt} - \mu \frac{d\sigma}{dt} - \frac{d\tau}{dt} = 0,
\end{aligned}$$

and, since $\nu = \rho\lambda + \sigma\mu + \tau$ is an integral hypersurface, this equation must be identically zero for all values of λ and μ , when we substitute for ν its value given by (7). We thus arrive at the following system of differential equations satisfied by the functions ρ , σ and τ :

$$\left. \begin{aligned}
\frac{d\rho}{dt} &= p_{13} + p_{12}\sigma + p_{14}\tau + \rho(p_{13}\rho - p_{23}\sigma + p_{34}\tau), \\
\frac{d\sigma}{dt} &= -p_{23} - p_{12}\rho + p_{24}\sigma + \sigma(p_{13}\rho - p_{23}\sigma + p_{34}\tau), \\
\frac{d\tau}{dt} &= p_{34} - p_{14}\rho - p_{24}\sigma + \tau(p_{13}\rho - p_{23}\sigma + p_{34}\tau),
\end{aligned} \right\} \quad (8)$$

a system analogous to (3).

Suppose now that we know a particular solution of (8); then the integral hypersurface is known, and we shall prove that *the system (3) can be reduced to a linear system in three variables*. Let the surface be written as before

$$\nu = \rho\lambda + \sigma\mu + \tau,$$

where ρ , σ , τ form a particular solution of (8). If we intersect this surface by the linear spaces $t = \text{const.}$, we get an infinity of planes which stand in a projective relation to each other by virtue of the transformation (4). We now introduce a projective transformation of coordinates, whereby the plane of each space $t = \text{const.}$ is transformed to infinity. Such a transformation is

$$\lambda_1 = \frac{\lambda}{\nu - \rho\lambda - \sigma\mu - \tau}, \quad \mu_1 = \frac{\mu}{\nu - \rho\lambda - \sigma\mu - \tau}, \quad \nu_1 = \frac{1}{\nu - \rho\lambda - \sigma\mu - \tau}. \quad (9)$$

Introducing these new variables in (3), taking also account of the equations (8), we get the following linear system:

$$\left. \begin{aligned}
\frac{d\lambda_1}{dt} &= -p_{13} - (p_{34}\tau + 2p_{13}\rho - p_{23}\sigma)\lambda_1 + (p_{12} - p_{13}\sigma)\mu_1 + (p_{14} - p_{13}\tau)\nu_1, \\
\frac{d\mu_1}{dt} &= p_{23} - (p_{12} - p_{23}\rho)\lambda_1 - (p_{24}\tau - 2p_{23}\sigma + p_{23}\rho)\mu_1 + (p_{24} + p_{23}\tau)\nu_1, \\
\frac{d\nu_1}{dt} &= -p_{34} - (p_{14} + p_{34}\rho)\lambda_1 - (p_{24} + p_{34}\sigma)\mu_1 - (p_{23}\rho + 2p_{34}\tau - p_{23}\sigma)\nu_1.
\end{aligned} \right\} \quad (10)$$

A knowledge of a particular solution of (8) has thus reduced the integration of (1) to the integration of the linear system (10), which we shall write in the general form

$$\left. \begin{aligned} \frac{d\lambda_1}{dt} &= a_1 + a_2\lambda_1 + a_3\mu_1 + a_4\nu_1, \\ \frac{d\mu_1}{dt} &= b_1 + b_2\lambda_1 + b_3\mu_1 + b_4\nu_1, \\ \frac{d\nu_1}{dt} &= c_1 + c_2\lambda_1 + c_3\mu_1 + c_4\nu_1. \end{aligned} \right\} \quad (11)$$

We now proceed in the same way as before, writing

$$\nu_1 = \rho_1\lambda_1 + \sigma_1\mu_1 + \tau_1,$$

where ρ_1, σ_1, τ_1 are unknown functions of t . Differentiating and substituting the values of $\frac{d\lambda_1}{dt}, \frac{d\mu_1}{dt}, \frac{d\nu_1}{dt}$, we get

$$\begin{aligned} c_1 + c_2\lambda + c_3\mu + c_4\nu - \rho_1(a_1 + a_2\lambda_1 + a_3\mu_1 + a_4\nu_1) \\ - \sigma_1(b_1 + b_2\lambda_1 + b_3\mu_1 + b_4\nu_1) - \lambda_1 \frac{d\rho_1}{dt} - \mu_1 \frac{d\sigma_1}{dt} - \frac{d\tau}{dt} = 0, \end{aligned}$$

and this equation must be identically zero for all values of λ_1 and μ_1 when we substitute for ν_1 its value. This gives us the following system of differential equations:

$$\left. \begin{aligned} \frac{d\rho_1}{dt} &= c_2 - (a_2 - c_4)\rho_1 - b_2\sigma_1 - \rho_1(a_4\rho_1 + b_4\sigma_1), \\ \frac{d\sigma_1}{dt} &= c_3 - a_3\rho_1 - (b_3 - c_4)\sigma_1 - \sigma_1(a_4\rho_1 + b_4\sigma_1), \\ \frac{d\tau_1}{dt} &= c_1 - a_1\rho_1 - b_1\sigma_1 - c_4\tau_1 - \tau_1(a_4\rho_1 + b_4\sigma_1). \end{aligned} \right\} \quad (12)$$

Let us suppose that the first two equations, which do not contain the variable τ_1 , have been integrated. Substituting the values of ρ_1 and σ_1 in the third it reduces to the form

$$\frac{d\tau_1}{dt} = A\tau_1 + B,$$

which may be integrated by two quadratures. The general integrals ρ_1, σ_1, τ_1 having been found, all the integral hypersurfaces $\nu_1 = \rho_1\lambda_1 + \sigma_1\mu_1 + \tau_1$ are known, and therefore also all the integral curves. We may therefore say: *If a particular solution of the system (8) is known, the integration of (3) is reduced to the integration of a system of generalized Riccati's equations in two variables and two quadratures.*

We may now proceed with the system (12), using a method identical with the one employed for three variables.* An integral surface may be written

$$\sigma_1 = \phi \rho_1 + \psi, \quad (13)$$

where ϕ and ψ are particular solutions of the equations

$$\begin{aligned} \frac{d\phi}{dt} &= -a_3 + (a_2 - b_3)\phi - a_4\psi - \phi(b_4\psi - b_2\phi), \\ \frac{d\psi}{dt} &= c_2 - c_2\phi - (b_3 - c_4)\psi - \phi(b_4\psi - b_2\phi). \end{aligned}$$

Employing the transformation

$$\rho'_1 = \frac{\rho_1}{\sigma_1 - \phi\rho_1 - \psi}, \quad \sigma'_1 = \frac{1}{\sigma_1 - \phi\rho_1 - \psi},$$

we finally arrive at the linear system

$$\begin{aligned} \frac{d\rho'_1}{dt} &= A_1\rho'_1 + B_1\sigma'_1 + C_1, \\ \frac{d\sigma'_1}{dt} &= A_2\rho'_1 + B_2\sigma'_1 + C_2, \end{aligned} \quad (14)$$

the integration of which can by a method identical with the one employed in the case of three variables be reduced to the integration of a Ricatti equation in a single variable and two quadratures. We may therefore say, *if a particular solution ρ, σ, τ of the system (8) and also a particular solution ϕ, ψ of the system (14) are known, the integration of the original system (1) is reduced to the integration of a Ricatti equation and four quadratures.*

II. We shall now consider the case where $\alpha, \beta, \gamma, \delta$ is a set of particular solutions satisfying the relation

$$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 0,$$

and let us suppose that the p 's are imaginary functions of t . In this case we are able to find a transformation by means of which the problem of integration is simplified a great deal.

We put

$$\left. \begin{aligned} \frac{\alpha + i\beta}{\gamma - i\delta} &= -\frac{(\gamma + i\delta)}{\alpha - i\beta} = x, \\ \frac{\alpha - i\beta}{\gamma - i\delta} &= -\frac{\gamma + i\delta}{\alpha + i\beta} = -\frac{1}{y}, \end{aligned} \right\} \quad (15)$$

* For a complete discussion of the generalized system of Ricatti's equations in two variables see Sophus Lie's "Vorlesungen über continuierliche Gruppen," p. 778. See also an article by Alf. Gulberg, "Zur Theorie der Differentialgleichungen," in Crelle's Journal, vol. 115.

which give us the relations

$$-i \frac{1-xy}{x-y} = \frac{\alpha}{\delta}, \quad \frac{1+xy}{x-y} = \frac{\beta}{\delta}, \quad i \frac{x+y}{x-y} = \frac{\gamma}{\delta},$$

or, introducing a factor of proportionality ρ ,

$$\alpha = -\rho i(1-xy), \quad \beta = \rho(1+xy), \quad \gamma = \rho i(x+y), \quad \delta = \rho(x-y).$$

Differentiating, we get

$$\begin{aligned} \frac{d\alpha}{dt} &= -i(1-xy) \frac{d\rho}{dt} + \rho i \left(y \frac{dx}{dt} + x \frac{dy}{dt} \right), \\ \frac{d\beta}{dt} &= (1+xy) \frac{d\rho}{dt} + \rho \left(y \frac{dx}{dt} + x \frac{dy}{dt} \right), \\ \frac{d\gamma}{dt} &= i(x+y) \frac{d\rho}{dt} + \rho i \left(\frac{dx}{dt} + \frac{dy}{dt} \right), \\ \frac{d\delta}{dt} &= (x-y) \frac{d\rho}{dt} + \rho \left(\frac{dx}{dt} - \frac{dy}{dt} \right). \end{aligned}$$

The first two of these equations give us at once

$$-2i \frac{d\rho}{dt} = \frac{d\alpha}{dt} - i \frac{d\beta}{dt}.$$

Substituting this value of $\frac{d\rho}{dt}$ in the last two equations and solving for $\frac{dx}{dt}$ and $\frac{dy}{dt}$, we get

$$\begin{aligned} 2i\rho \frac{dx}{dt} &= \frac{d\gamma}{dt} + i \frac{d\delta}{dt} + x \left(\frac{d\alpha}{dt} - i \frac{d\beta}{dt} \right), \\ 2i\rho \frac{dy}{dt} &= \frac{d\gamma}{dt} - i \frac{d\delta}{dt} + y \left(\frac{d\alpha}{dt} - i \frac{d\beta}{dt} \right). \end{aligned}$$

Introducing on the right-hand side the variables x , y and ρ and reducing we get the following equations:

$$\left. \begin{aligned} \frac{dx}{dt} &= -i(p_{12} - p_{34})x + \frac{(p_{14} + p_{23})i - (p_{13} + p_{24})}{2} \\ &\quad + \frac{-(p_{13} + p_{24}) - (p_{14} + p_{23})i}{2} x^2, \\ \frac{dy}{dt} &= -i(p_{12} - p_{34})y + \frac{(p_{23} - p_{14})i - (p_{13} - p_{24})}{2} \\ &\quad + \frac{-(p_{13} - p_{24}) - (p_{23} - p_{14})i}{2} y^2, \\ \frac{i}{\rho} \frac{d\rho}{dt} &= -p_{12} + \frac{(p_{24} + p_{13})i - (p_{14} + p_{23})}{2} x + \frac{p_{14} - p_{23} + (p_{13} - p_{24})i}{2} y. \end{aligned} \right\} \quad (16)$$

The equations in x and y have the well-known Ricatti's form, and resemble those deduced by Darboux in his "Leçons sur la Théorie des Surfaces," vol. I, p. 22. We now put

$$u = x + \frac{1}{\lambda}, \quad v = y + \frac{1}{\mu},$$

the above-mentioned equations will take the form

$$\begin{aligned} \frac{d\lambda}{dt} = & \frac{-(p_{13} + p_{24}) - (p_{14} + p_{23})i}{2} \\ & + [i(p_{12} - p_{34}) + \{(p_{13} + p_{24}) + (p_{14} + p_{23})i\}x]\lambda, \\ \frac{d\mu}{dt} = & \frac{-(p_{13} - p_{24}) - (p_{23} - p_{14})i}{2} \\ & + [i(p_{12} - p_{34}) + \{(p_{13} - p_{24}) + (p_{23} - p_{14})i\}y]\mu, \end{aligned}$$

which may be solved by means of 4 quadratures. We have thus arrived at the following result: *If $\alpha, \beta, \gamma, \delta$ be a particular solution of (1) satisfying the relation $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 0$, and if also the p 's are imaginary functions of t , then four quadratures will be necessary for the complete integration of (1).*

Suppose that the p 's are real; if x be a particular solution of the equation in x , then will $-\frac{1}{x'}$, where x' denotes the conjugate imaginary of x , also be a particular solution of the same equation. This is easily seen by changing i into $-i$ on the right-hand side and putting x' for x . (Compare Darboux, "Leçons," p. 23.) The same holds for the equation in y , so that we get two pairs of particular solutions $x, -\frac{1}{x'}$; $y, -\frac{1}{y'}$. But we know that if two solutions of a Ricatti equation are known, only one quadrature is necessary for the complete integration of the equation. It follows therefore that when the p 's are real functions of t , only two quadratures are necessary for the complete integration of (1), $\alpha, \beta, \gamma, \delta$ being a given particular solution.